Manifest covariance and the Hamiltonian approach to mass gap in (2+1)-dimensional Yang-Mills theory

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Abstract

In earlier work we have given a Hamiltonian analysis of Yang-Mills theory in (2+1) dimensions showing how a mass gap could arise. In this paper, generalizing and covariantizing from the mass term in the Hamiltonian analysis, we obtain two manifestly covariant and gauge-invariant mass terms which can be used in a resummation of standard perturbation theory to study properties of the mass gap.

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1. Introduction

In a series of recent papers we have carried out a Hamiltonian analysis of Yang-Mills theories in (2+1) dimensions, YM_{2+1} [1, 2, 3]. A matrix parametrization of the gauge potentials A_μ was used which facilitated calculations using manifestly gauge-invariant variables. An analytical formula for the string tension was obtained which was found to be in good agreement with lattice gauge theory simulations [3, 4]. It was also shown that effectively the gauge bosons become massive. This mass can be identified in the context of a (3+1)-dimensional gluon plasma as the magnetic mass [5]. The analytically calculated value of this mass is also in reasonable agreement with numerical estimates [6].

All the above calculations were done in a Hamiltonian framework. The virtue of this approach is that at a given time we have to consider gauge potentials on the two-dimensional space and for two-dimensional gauge fields a number of calculations can be done exactly. However, as in any Hamiltonian analysis, we do not have manifest Lorentz covariance. Overall Lorentz covariance is not lost since the requisite commutation properties on the components of the energy-momentum tensor may be verified [1]. Now, the main physical context in which our results could be applied would be the case of magnetic screening in QCD at high temperatures. The Wick-rotated version of YM_{2+1}, namely three-dimensional Euclidean Yang-Mills theory, is what is needed to describe the zero Matsubara frequency mode of the (3+1)-dimensional QCD at high temperatures. A manifestly covariant formulation of our analysis would be just what is ideal in relating our results to Feynman diagrams in high temperature QCD. There are two sources of lack of manifest covariance in our approach, firstly due to the use of the Hamiltonian analysis itself and secondly, because the gauge-invariant variables we used were defined intrinsically in a (2+1)-splitting and do not have simple (tensorial) transformation properties under Lorentz transformations. Going over to a Lagrangian might address the first problem of degrees of freedom being defined at a constant time but not the second, unless we have a Lorentz covariant parametrization of the gauge potentials which makes it easy to isolate the gauge-invariant degrees of freedom. In our approach, calculability, viz., the fact that the transformation of variables could be done exactly, including the Jacobian, was the crucial factor, which led to physical results. To be useful to a similar degree, one needs a Lorentz covariant parametrization of A_μ for which the change of variables to the gauge-invariant degrees of freedom can be carried out, including the path-integral Jacobian in a nonperturbative way. We have not been able to find such a set of variables. The situation is similar to the old problem of rewriting Yang-Mills theory in terms of Wilson loop variables and other similar choices of variables; as in many earlier attempts, the technical stumbling block has been the calculation of Jacobians in nonperturbative terms.

A more practical alternative strategy would then be the following. First of all, we can consider an expansion of our results in powers of the coupling constant. It then becomes clear that the mass gap cannot be seen to any finite order in the perturbative expansion but could be obtained by resummation of certain series of terms. Such resummations can be carried out in the covariant path-integral approach by adding and subtracting suitable (gauge-invariant) mass terms, and indeed, many such calculations have already been done using different choices of mass terms [7, 8, 9]. In these calculations, there is no unique or preferred mass term we can use. The natural question is whether our Hamiltonian analysis can shed any light on this issue; in other words, are there any mass terms which are similar or close to the mass term which arises in the Hamiltonian analysis?

In this paper we do the following. We study the properties of the mass term which arises in our Hamiltonian analysis, identifying certain key features and then seek covariant gauge-invariant mass terms which can be used in a Lagrangian resummation procedure and which are simple generalizations of what we find in the Hamiltonian analysis. Two such terms are considered and analyzed to some extent.

In the next section, we discuss an "improved" version of perturbation theory starting with our Hamiltonian analysis. We first show how the mass term can be manifestly displayed to the lowest order in our gauge-invariant variables. Then building upon this lowest order result, we identify the required properties and the nature of the mass term. A procedure for the covariantization of the mass term is decribed in section 3. Explicit formulae for the covariantized mass terms are given to cubic order in the potentials. Section 4 gives a brief discussion of the results of carrying out the resummation to the lowest nontrivial order. The paper concludes with a short summarizing discussion. Some technical arguments on the nature of the mass term are given in the appendix.

2. "Improved" perturbation theory and the mass term

We consider an $SU(N)$-gauge theory with the gauge potentials $A_i = -it^a A_i^a$, $i = 1, 2$, where t^a are hermitian $(N \times N)$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc}t^c$, $\mathrm{Tr}(t^a t^b) = \frac{1}{2}\delta^{ab}$. The Hamiltonian analysis was carried out in the $A_0 = 0$ gauge with the spatial components of the gauge potentials parametrized as

$$A = -\partial M M^{-1}, \qquad\qquad \bar{A} = M^{\dagger -1}\bar{\partial} M^{\dagger} \tag{1}$$

Here $A = \frac{1}{2}(A_1 + iA_2)$, $\bar{A} = \frac{1}{2}(A_1 - iA_2)$, $z = x_1 - ix_2$, $\bar{z} = x_1 + ix_2$, $\partial = \frac{1}{2}(\partial_1 + i\partial_2)$, $\bar{\partial} = \frac{1}{2}(\partial_1 - i\partial_2)$. In the above equation, M, $M^{\dagger}$ are complex $SL(N, \mathbf{C})$-matrices. The volume element on the space $\mathcal{C}$ of gauge-invariant configurations was calculated explicitly in [1,2] and found to be

$$d\mu(\mathcal{C}) \;=\; \frac{[dA d\bar{A}]}{\mathrm{vol}\mathcal{G}} \;=\; d\mu(H)\; e^{2c_A I(H)} \tag{2}$$

where $H = M^{\dagger}M$. H is a gauge-invariant, hermitian matrix-valued field. $d\mu(H)$ is the Haar measure for H. (Explicitly, it may be written as $d\mu(H) = [d\varphi^a] \prod_x \det r$ where $H^{-1}dH = d\varphi^a r_{ak}(\varphi) t_k$.) c_A is the quadratic Casimir of the adjoint representation, $c_A \delta^{ab} = f^{amn} f^{bmn}$. $I(H)$ is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field H given by [10]

$$I(H) = \frac{1}{2\pi}\int \mathrm{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi}\int \epsilon^{\mu\nu\alpha}\mathrm{Tr}(H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\alpha H) \tag{3}$$

As is typical for the WZW action, the second integral is over a three-dimensional space whose boundary is the physical two-dimensional space corresponding to the coordinates $z,\ \bar{z}$. The integrand thus requires an extension of the matrix field H into the interior of the three-dimensional space, but physical results do not depend on how this extension is done [10]. Actually for the special case of hermitian matrices, the second term can also be written as an integral over spatial coordinates only [11].

The inner product for two wavefunctions $\Psi_1,\ \Psi_2$ is given by

$$\langle 1|2\rangle = \int d\mu(\mathcal{C})\ \Psi_1^*(H)\Psi_2(H) = \int d\mu(H)\ e^{2c_A I(H)}\ \Psi_1^*(H)\Psi_2(H) \tag{4}$$

Carrying out the change of variables from A to H in the Hamiltonian operator, one gets

$$\begin{aligned}
\mathcal{H} &= T + V \\
T &= \frac{e^2 c_A}{2\pi}\left[\int_u J^a(\vec{u})\frac{\delta}{\delta J^a(\vec{u})} \ + \ \int \Omega^{ab}(\vec{u},\vec{v})\frac{\delta}{\delta J^a(\vec{u})}\frac{\delta}{\delta J^b(\vec{v})}\right] \\
V &= \frac{\pi}{m c_A}\int \bar{\partial} J_a \bar{\partial} J_a \\
J &= \frac{c_A}{\pi}\partial H H^{-1} \\
\Omega^{ab}(\vec{u},\vec{v}) &= \frac{c_A}{\pi^2}\frac{\delta^{ab}}{(u-v)^2} \ - \ i\frac{f^{abc}J^c(\vec{v})}{\pi(u-v)}
\end{aligned} \tag{5}$$

The first term in the kinetic energy T, viz., $J(\delta/\delta J)$ shows that every power of J in the wavefunction will give a contribution $m = e^2 c_A/2\pi$ to the energy. This is the basic mass gap of the theory.

The volume element (2) plays a crucial role in how the theory develops a mass gap. If $I(H)$ is expanded in powers of the magnetic field $B^a = \frac{1}{2}\epsilon_{ij}(\partial_i A_j^a - \partial_j A_i^a + f^{abc}A_i^b A_j^c)$, the leading term has the form

$$I(H) \approx \frac{1}{4\pi}\int B\left(\frac{1}{\nabla^2}\right)B + \mathcal{O}(B^3) \tag{6}$$

Writing $\Delta E,\ \Delta B$ for the root mean square fluctuations of the electric field E and the magnetic field B, we have, from the canonical commutation rules $[E_i^a, A_j^b] = -i\delta_{ij}\delta^{ab}$, $\Delta E\ \Delta B \sim k$, where k is the momentum variable. This gives an estimate for the energy

$$\mathcal{E} = \frac{1}{2}\left(\frac{e^2 k^2}{\Delta B^2} + \frac{\Delta B^2}{e^2}\right) \tag{7}$$

For low lying states, we must minimize $\mathcal{E}$ with respect to ΔB^2, $\Delta B^2_{min} \sim e^2 k$, giving $\mathcal{E} \sim k$. This corresponds to the standard photon. For the nonabelian theory, this is inadequate since $\langle\mathcal{H}\rangle$ involves the factor $e^{2c_A I(H)}$. In fact,

$$\langle\mathcal{H}\rangle \ = \int d\mu(H)\ e^{2c_A I(H)}\ \tfrac{1}{2}(e^2E^2 + B^2/e^2) \tag{8}$$

Equation (6) shows that B follows a Gaussian distribution of width $\Delta B^2 \approx \pi k^2/c_A$ for small values of k. This Gaussian dominates near small k giving $\Delta B^2 \sim k^2(\pi/c_A)$. In other words, even though $\mathcal{E}$ is minimized around $\Delta B^2 \sim k$, probability is concentrated around $\Delta B^2 \sim k^2(\pi/c_A)$. For the expectation value of the energy, we then find $\mathcal{E} \sim e^2 c_A/2\pi + \mathcal{O}(k^2)$. Thus the kinetic term in combination with the measure factor $e^{2c_A I(H)}$ could lead to a mass gap of order $e^2 c_A$. The argument is not rigorous, but captures the essence of how a mass gap arises in our formalism [1].

All we have done so far is to rewrite the theory in terms of gauge-invariant variables without making any other approximation. It is therefore possible to look at perturbation theory in this version. Since c_A is quadratic in the structure constants f^{abc}, the exponent in (2) would be considered a second order effect in the perturbative expansion. The exponential in (2) would be expanded in powers of c_A and we would not see a Gaussian distribution for the magnetic fluctuations (of width $\sim k^2$). Hence the effect considered above cannot be seen to any finite order. The basic question we are asking in this paper is whether one can incorporate the effects of the nontrivial measure (2) and the resultant mass term in a covariant path integral for diagrammatic analysis. It is clear that this cannot be done at any finite order in perturbation theory. However, one can define an "improved" perturbation theory where a partial resummation of the perturbative expansion has been carried out [2]. This improvement would be equivalent to keeping the leading term of $I(H)$ as in (6) in the exponent in (2). For example, if we write $H = e^{t^a\varphi^a} \approx 1 + t^a\varphi^a$, as would be appropriate in perturbation theory, we find

$$d\mu(\mathcal{C}) \simeq [d\varphi] e^{-\frac{c_A}{2\pi}\int \partial\varphi^a \bar{\partial}\varphi^a} \Big(1 \ + \mathcal{O}(\varphi^3)\Big) \tag{9}$$

Correspondingly, $J^a \simeq \frac{c_A}{\pi}\partial\varphi^a$, and the Hamiltonian has the expansion

$$\mathcal{H} \simeq m\left[\int \varphi_a \frac{\delta}{\delta\varphi_a} + \frac{\pi}{c_A}\int \Omega(\vec{x},\vec{y}) \frac{\delta}{\delta\varphi_a(\vec{x})}\frac{\delta}{\delta\varphi_a(\vec{y})}\right] + \frac{c_A}{m\pi}\int \partial\varphi_a(-\partial\bar{\partial})\bar{\partial}\varphi_a \ + \mathcal{O}(\varphi^3) \tag{10}$$

where $m = e^2 c_A/2\pi$ and $\Omega(\vec{x},\vec{y}) = -\int \frac{d^2k}{(2\pi)^2}\ e^{ik\cdot(x-y)}/k\bar{k}$. The term $\int \varphi_a \delta/\delta\varphi_a$ shows that every φ in a wavefunction would get a contribution m to the energy; this is essentially the mass gap again.

The mass term can also be written in a different way as follows. We can absorb the exponential factor of (9) into the wavefunctions, defining $\Phi = e^{-\frac{c_A}{4\pi}\int \partial\varphi\bar{\partial}\varphi}\Psi$, so that the norm of Φ 's involves just integration of $\Phi^*\Phi$ with the flat measure $[d\varphi]$, i.e.,

$$\langle 1|2\rangle \approx \int [d\varphi]\ \ \Phi_1^*(H)\Phi_2(H) \tag{11}$$

For the wavefunctions Φ, we get

$$\begin{aligned} \mathcal{H}' \quad &\simeq \frac{1}{2}\int_x \left[-\frac{\delta^2}{\delta\phi_a^2(\vec{x})} + \phi_a(\vec{x})(m^2 - \nabla^2)\phi_a(\vec{x})\right] + \ldots \\ &\simeq \frac{1}{2}\int_x \left[-\frac{\delta^2}{\delta\phi_a^2(\vec{x})} + \phi_a(\vec{x})(-\nabla^2)\phi_a(\vec{x}) + \frac{e^2 c_A^2}{4\pi^2}\ \partial\varphi_a\bar{\partial}\varphi_a\right] \end{aligned} \tag{12}$$

where $\phi_a(\vec{k}) = \sqrt{c_A k\bar{k}/(2\pi m)}\ \varphi_a(\vec{k})$. Expression (12) is the Hamiltonian for a field of mass $m = e^2 c_A/2\pi$. This can be taken as the lowest order term of an "improved" perturbation theory. In the second line of (12), we have also separately shown the mass term since we shall need it shortly.

It may be worth emphasizing that this Hamiltonian (12), with the inner product (11), is entirely equivalent to the previous one (10), with the inner product given by (9) [12]. However, in (12), the mass term has a more conventional form and therefore one can use this as a starting point for the mass term we want to find for resummation calculations in the Lagrangian formalism. We also see that the energy of the particle, viz., $\sqrt{k^2+m^2}$ is an infinite series when expanded in powers of e^2. The "improved" perturbation theory, which is effectively resumming this up, is thus equivalent to a partial resummation of the perturbative expansion.

The gauge-invariant variables φ_a or H are wonderfully appropriate for the Hamiltonian analysis. However, in a perturbative diagrammatic calculation carried out in a covariant Lagrangian framework, we shall need to use the gauge potentials A_i. To the lowest order, the number of powers of φ's and A_i's do match; the mass term given in (12) is thus quadratic in the A's and can be written as

$$\frac{e^2 c_A^2}{4\pi^2} \int \partial\varphi_a \bar{\partial}\varphi_a = \frac{m^2}{e^2} \int \frac{d^2k}{(2\pi)^2} A_i^a(-k) \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) A_j^a(k) \tag{13}$$

This gives the mass term only to the quadratic order and does not have the full nonabelian gauge invariance; there will be terms with higher powers of A's giving a gauge-invariant completion of (13). Already at this stage we can say something about how the full mass term should look like, based on the following conditions.

1. The mass term F should be expressible in terms of H since that is the basic gauge-invariant variable of the theory. (The φ_a's represent a particular way to parametrize H. It should be possible to write the mass term in a way that is not sensitive to how we parametrize H.)

2. To the lowest, viz., quadratic order, it should agree with the mass term in (12) or (13).

3. The mass term should have "holomorphic invariance".

The last property is the following requirement. As can be seen from the definitions (1), the matrices $(M,\ M^\dagger)$ and $(M\bar{V}(\bar{z}),\ V(z)M^\dagger)$ both define the same potentials $(A, \bar{A})$, where $V(z)$ is holomorphic in z and $\bar{V}(\bar{z})$ is antiholomorphic. In terms of H, this means that H and $VH\bar{V}$ are physically equivalent. Physical quantities should be, and in any correct calculation will be, invariant under $H \to VH\bar{V}$, so that the ambiguity in the choice of the matrices $M,\ M^\dagger$ does not affect the physics. For example, the WZW action in (3) is invariant under $H \to VH\bar{V}$, a property familiar from two-dimensional physics. We have previously referred to this invariance requirement as "holomorphic invariance"; it can be used as a guide in some calculations.

A minimal mass term with the above requirements can be easily written down. First of all, since $H = e^{t^a \varphi^a}$, we see that, in terms of H, the mass term shown in (12) is of the form $\mathrm{Tr}(\partial H \bar{\partial} H^{-1})$. (We shall discuss this in the appendix in some detail. The key point is that we have $\mathrm{Tr}(\partial H \bar{\partial} H^{-1})$ and not something like $\mathrm{Tr}(\partial H \bar{\partial} H)$, eventhough the latter does have the same kind of quadratic approximation.) Notice that this term, $\mathrm{Tr}(\partial H \bar{\partial} H^{-1})$, is the first term of the WZW action (3). Since the WZW action has holomorphic invariance, we see that a minimal mass term, or a minimal holomorphically invariant completion of $\mathrm{Tr}(\partial H \bar{\partial} H^{-1})$, with the properties 1-3 listed above is also a WZW action, i.e.,

$$F_{min} = -\frac{2\pi}{e^2} I(H) \tag{14}$$

Of course, one can always add gauge-invariant terms which start with cubic or higher powers of A, which do not spoil the requirement that it agrees with (12) at the quadratic order. In this sense the WZW action is only a minimal mass term, not unique. (The quadratic part is, of course, unique.)

There are also other invariant ways to complete $\mathrm{Tr}(\partial H \bar{\partial} H^{-1})$. For example, we can write

$$F = \frac{\pi^2}{2e^2 c_A^2} \int (\bar{G} \bar{\partial} J^a) H_{ab} (G \partial \bar{J}^b) \tag{15}$$

where $J^a = (c_A/\pi)(\partial H H^{-1})^a$, $\bar{J}^a = (c_A/\pi)(H^{-1} \bar{\partial} H)^a$ and $H_{ab} = 2\mathrm{Tr}(t_a H t_b H^{-1})$. This will be holomorphically invariant with the Green's functions $G = \partial^{-1}$ and $\bar{G} = \bar{\partial}^{-1}$ transforming in a certain way as discussed in [2]. This way of writing F involves the additional use of Green's functions, over and above the Green's functions which appear in the construction of H (or M, $M^\dagger$) from the potentials. In the next section, we give expressions for the covariantized versions of both F_{min} of (14) and F as in (15), to cubic order in potentials. We shall see that F equals F_{min} plus a number of terms which involve the logarithms of momenta, the latter having to do with the additional Green's functions. *The only holomorphically invariant completion of* $\mathrm{Tr}(\partial H \bar{\partial} H^{-1})$ *using H and its derivatives, but no additional Green's functions, is $I(H)$ as given in (14).* This is why we refer to it as the minimal term.

A strategy of doing the resummation calculations is then to consider the action

$$S = S_{YM} - \mu^2 \ F_{min} \ + \Delta \ F_{min} \tag{16}$$

where we consider Δ to be of one higher order in a loop expansion compared to μ^2 and S_{YM} is the usual action for the YM path integral. In other words, the loop expansion is organized by treating $S_{YM} - \mu^2 F_{min}$ as the zeroth order term, while ΔF_{min} contributes at one loop higher. In particular Δ is a parameter which is taken to have a loop expansion, viz., $\Delta = \Delta^{(1)} + \Delta^{(2)} + ...$ Since the parameter μ^2 is still arbitrary, we can choose it to be the exact value of the pole of the full propagator. In other words, the pole of the propagator (for the transverse potentials) remains μ^2 as loop corrections are added. This requires choosing $\Delta^{(1)}$ to cancel the one-loop shift of the pole, $\Delta^{(2)}$ to cancel the two-loop shift of the pole, etc. $\Delta^{(1)}$, $\Delta^{(2)}$, etc. are calculated as functions of the parameter μ^2. The condition $\mu^2 = \Delta$ then becomes a nonlinear equation for μ^2; it is the gap equation given as

$$\Delta(\mu) = \Delta^{(1)} + \Delta^{(2)} + ... = \mu^2 \tag{17}$$

This determines μ to the order to which the calculation is performed. Thus in the end we also have $\mu^2 = \Delta$ as desired. One can do similar resummation and gap equations wth any mass term, for example F in place of F_{min} in (16).

This procedure is, of course, what is done in any kind of resummation or gap equation approach to mass generation [7, 8, 9]. The additional ingredient for us is that the Hamiltonian analysis suggests some specific forms of the mass term (14). The mass terms (14, 15) are not covariant, so we have to write covariantized versions of these before they can be used in a covariant resummation calculation. We shall now consider a procedure for covariantization, which is of interest in its own right.

3. Covariantization of the mass term

General procedure

There is one more problem we have to deal with in using $I(H) = I(A,\ \bar{A})$ in a resummed perturbation theory, namely, that it is not manifestly covariant. Again, the original theory is Lorentz invariant and adding and subtracting I does not affect this. However when we take μ^2 and Δ to be of different orders, we lose covariance order-by-order unless we use a covariantized version of I. In this section we outline a general method of covariantization which can be used for F, F_{min}. Our method may also be interesting in its own right.

The key expressions we have involve holomorphic and antiholomorphic derivatives and fields. We observe that $\partial = \frac{1}{2} n_o^a \partial_a$ and $\bar{\partial} = \frac{1}{2} \bar{n}_o^a \partial_a$, where $n_o^a = (1, i, 0)$ and $\bar{n}_o^a = (1, -i, 0)$. Similarly for the gauge fields A, $\bar{A}$. An arbitrary Lorentz transformation of n_o^a and $\bar{n}_o^a$ produces null 3-vectors n^a, $\bar{n}^a$ respectively, such that

$$\begin{aligned} n^a n_a &= g_{ab} n^a n^b = 0 \\ \bar{n}^a \bar{n}_a &= g_{ab} \bar{n}^a \bar{n}^b = 0 \\ n^a \bar{n}_a &= g_{ab} n^a \bar{n}^b = 2 \end{aligned} \tag{18}$$

where g_{ab} is the Minkowski metric. We shall consider the signature (1,1,-1). This suggests the following covariantization procedure. Replace the holomorphic and antiholomorphic derivatives $(\partial,\ \bar{\partial})$ and gauge fields $(A,\ \bar{A})$ in $I(H)$, expressed in terms of the potentials, by $(\frac{1}{2} n \cdot \partial,\ \ \frac{1}{2} \bar{n} \cdot \partial)$ and $(\frac{1}{2} n \cdot A,\ \ \frac{1}{2} \bar{n} \cdot A)$ respectively, and then integrate over Lorentz transformations. Thus the covariant analogue of a general term

$$S = \int dt d^2x\ \mathcal{L}(A, \bar{A}, \partial, \bar{\partial}) \tag{19}$$

would be

$$S_{cov} = \int d\mu \int dt d^2x\ \mathcal{L}(\tfrac{1}{2} n \cdot A,\ \tfrac{1}{2} \bar{n} \cdot A,\ \tfrac{1}{2} n \cdot \partial,\ \tfrac{1}{2} \bar{n} \cdot \partial) \tag{20}$$

where $d\mu$ is the measure over Lorentz transformations.

A particular parametrization for $n,\ \bar{n}$ is given by

$$\begin{aligned} n_a &= (\cosh\theta\cos\chi - i\sin\chi,\ \ \cosh\theta\sin\chi + i\cos\chi,\ \ \sinh\theta) \\ \bar{n}_a &= (\cosh\theta\cos\chi + i\sin\chi,\ \ \cosh\theta\sin\chi - i\cos\chi,\ \ \sinh\theta) \end{aligned} \tag{21}$$

In terms of this parametrization, $d\mu = d(\cosh\theta)d\chi$, where $\cosh\theta \in (0,\infty)$ and $\chi \in (0, 2\pi)$.

The problem with this procedure is the fact that the Lorentz group is noncompact and integration over Lorentz transformations leads to divergences. The degree of divergence depends on the number of n's and $\bar{n}$'s in the integrand. In order then for the covariantization procedure to be meaningful one needs to regulate the integrals in a consistent way. As we show below, this can be done by replacing the integrals by traces of suitable $(M \times M)$-matrices. The integrals are then regained in a large M-limit. To define the regularization, notice first of all that there is no such problem in Euclidean three-dimensional space. Integration over Lorentz transformations is replaced by integration over rotation angles and is convergent. This has been used before in constructing covariant mass terms in Euclidean space [13, 7]. The Euclidean version of the null vectors n is

$$\begin{aligned} n_i &= (-\cos\theta\cos\chi - i\sin\chi,\ \ -\cos\theta\sin\chi + i\cos\chi,\ \ \sin\theta) \\ \bar{n}_i &= (-\cos\theta\cos\chi + i\sin\chi,\ \ -\cos\theta\sin\chi - i\cos\chi,\ \ \sin\theta) \end{aligned} \tag{22}$$

The measure of integration over the angles is $d\Omega = \sin\theta d\theta d\chi$. The Euclidean vectors $n,\ \bar{n}$ obey the same properties (18), but with the Minkowski metric replaced by the Euclidean one. If this procedure is used for the minimal mass term $F_{min} = -(2\pi/e^2)I(H)$, the resulting covariant mass term is precisely what was proposed some time ago in [13] and used in [7]. It is interesting that this mass term emerges in some minimal way from our Hamiltonian analysis.

The Euclidean analysis is adequate for diagrammatic calculations. However, conceptually, there is still something lacking. Hamiltonian analysis is all in Minkowski space and to tie in everything, it is important to define the covariantization directly in Minkowski space as well. In view of the Euclidean result, one way to define the regularization of the integration over the Lorentz transformations is then as follows. We do a Wick rotation of the integrands to Euclidean space, do the integrals there and then continue the final results back to Minkowski space. Alternatively, one can seek a definition of the regularized integrals in Minkowski space directly in such a way that the results agree with the Wick rotation of the Euclidean results. We now show how this can be done.

First we construct the operator analogues of the Minkowski null vectors $n^a,\ \bar{n}^a$. Let g be a group element of $SO(2,1)$. g can be written as $g = e^{it^a\theta^a}$ where

$$t^a = (i\sigma_1, i\sigma_2, \sigma_3) \tag{23}$$

and σ_a, $a = 1, 2, 3$, are the Pauli matrices. The matrices t^a satisfy the commutation rules

$$[t^a,\ t^b] = 2i\ \epsilon^{abc} g_{cd}\ t^d \tag{24}$$

We now introduce the operators $a,\ a^\dagger$, which are doublets under $SO(2,1)$. One can show that the generators of $SO(2,1)$ can be written as

$$J^a = \bar{a}\ \frac{t^a}{2}\ a \tag{25}$$

where $\bar{a} = a^\dagger \sigma_3$. The commutation rule for $a,\ a^\dagger$, compatible with $SO(2,1)$ invariance, is

$$[a_i, \bar{a}_j] = \delta_{ij}, \qquad\qquad i,j = 1,2 \tag{26}$$

We now define the following operators

$$\begin{aligned} S^a &= \bar{a}\ t^a t^2\ \bar{a}^T \\ \bar{S}^a = (S^a)^\dagger &= a^T\ t^2 t^a\ a \end{aligned} \tag{27}$$

(the superscript T indicates the transpose.) It is easy to show that both S and $\bar{S}$ transform as vectors under $SO(2,1)$ transformations. Further they are null vectors,

$$S^a S_a = 0\ \ , \qquad\qquad \bar{S}^a \bar{S}_a = 0 \tag{28}$$

and

$$S^a \bar{S}_a = 2(Q^2 - Q) \tag{29}$$

where $Q = \sum_{i=1}^{2} \bar{a}_i a_i$. Q is invariant under $SO(2,1)$ transformations.

The commutation relation between S and $\bar{S}$ is given by

$$[S^a,\ \bar{S}^b] = -2g^{ab}(2Q+2) + 8i\epsilon^{abc} g_{cd}\ J^d \tag{30}$$

In showing (28, 29, 30) the following properties of the t-matrices were used

$$\begin{aligned} (t^a)_{ij}(t_a)_{kl} &= -(2\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}) \\ t^a t^b &= -g^{ab} + i\epsilon^{abc} g_{cd} t^d \end{aligned} \tag{31}$$

Finite dimensional representations of $SO(2,1)$ may be constructed in terms of Fock states built up using $\bar{a}$ acting on a vacuum state, with a fixed value of Q, say, $M-1$. A basis of such states is given by $|r,s\rangle = C^{-1} \bar{a}_1^r \bar{a}_2^s |0\rangle$ with $r+s = M-1$ and $C = \sqrt{r!s!}$. There are M such states and matrix elements of J^a between these states will give the $(M \times M)$-matrix representation of $SO(2,1)$. We are interested in the action of $S^a,\ \bar{S}^a$ (or functions of these) on the states of $|r,s\rangle$ of this M-dimensional representation. In this case, we introduce the rescaled operators

$$\tilde{S}^a = \frac{S^a}{M}\ \ , \qquad \tilde{\bar{S}}^a = \frac{\bar{S}^a}{M} \tag{32}$$

In the large M-limit, as $M \to \infty$, the operators $\tilde{S},\ \tilde{\bar{S}}$ commute,

$$[\tilde{S}^a,\ \tilde{\bar{S}}^b] = 0 \tag{33}$$

Further

$$\tilde{S}^a \tilde{S}_a = 0 \quad , \quad \bar{\tilde{S}}^a \bar{\tilde{S}}_a = 0$$
$$\tilde{S}^a \bar{\tilde{S}}_a = 2$$

These properties are just what we have for $n,\ \bar{n}$ and so we can identify $\tilde{S},\ \bar{\tilde{S}}$ with $n, \bar{n}$, in the large M-limit.

We are interested in operators F made up of equal numbers of $\tilde{S}$ and $\bar{\tilde{S}}$'s. The trace of such an operator over states of fixed $Q = M - 1$ can be written as

$$\mathrm{Tr}\ F = \sum_{r,s=0}^{M-1} \langle r, s|F|r, s\rangle \tag{34}$$

Since $\tilde{S}$'s and $\bar{\tilde{S}}$'s are vectors of $SO(2,1)$, their traces have to produce invariant tensors of $SO(2,1)$. In the large M-limit, replacing $\tilde{S},\ \bar{\tilde{S}}$ by $n,\ \bar{n}$, F becomes a function of $n,\ \bar{n}$ and trace can be identified as integration. Further, noting that the trace of identity is M, we can define the regularized notion of integrals of products of $n,\ \bar{n}$ over the Lorentz group as

$$\left[\int d\mu F(n,\ \bar{n})\right]_{reg} = \frac{1}{M}\mathrm{Tr}\ F(\tilde{S},\ \bar{\tilde{S}})\Big]_{M\to\infty} \tag{35}$$

As an example of this definition of regularized integrals, we shall evaluate the integrals of $n^a\bar{n}^b$ and $n^a n^b \bar{n}^c \bar{n}^d$. According to our regularization prescription

$$\left[\int d\mu\ n^a\bar{n}^b\right]_{\mathrm{reg}} = \frac{1}{M^3}\mathrm{Tr}(S^a\bar{S}^b)\Big]_{M\to\infty} \tag{36}$$

Using the definition of $S^a,\ \bar{S}^b$ in (27) and the properties (31) we find that

$$\left[\int d\mu\ n^a\bar{n}^b\right]_{\mathrm{reg}} = \frac{2}{3}\ g^{ab} \tag{37}$$

The same result can be obtained more efficiently by using the fact that $\mathrm{Tr}(S^a\bar{S}^b)$ has to be proportional to the invariant tensor g^{ab},

$$\frac{1}{M^3}\mathrm{Tr}(S^a\bar{S}^b) = x\ g^{ab} \tag{38}$$

The constant of proportionality x is determined by multiplying both sides by g^{ab} and using the property (29). Similarly we can evaluate

$$\left[\int d\mu\ n^a n^b\bar{n}^c\bar{n}^d\right]_{\mathrm{reg}} = \frac{1}{M^5}\mathrm{Tr}(S^a S^b\bar{S}^c\bar{S}^d)\Big]_{M\to\infty}$$

$$= -\frac{4}{15}g^{ab}g^{cd} + \frac{4}{10}(g^{ac}g^{bd} + g^{ad}g^{bc}) \tag{39}$$

The Euclidean integrals corresponding to the above expressions can be calculated directly and one can verify that their Wick rotations agree with the above. In other words, we have the result

$$\left[\frac{1}{M}\mathrm{Tr}F(\tilde{S},\bar{\tilde{S}})\right]_{M\to\infty} = \text{Wick rotation of } \left[\int\frac{d\Omega}{4\pi}F(n,\bar{n})\right]_{Euclidean} \tag{40}$$

Given the above procedure of covariantization, we can write down the covariant version of the mass term (14). We generalize the derivatives and potentials appearing in $I(H)$ by defining $\partial = \frac{1}{2}\tilde{S}\cdot\partial,\ \bar{\partial} = \frac{1}{2}\bar{\tilde{S}}\cdot\partial$ and $A = \frac{1}{2}\tilde{S}\cdot A,\ \bar{A} = \frac{1}{2}\bar{\tilde{S}}\cdot A$. The minimal covariant mass term may now be obtained as

$$F_{min} = \int\ \frac{1}{M}\mathrm{Tr}\left\{-\frac{2\pi}{e^2}I(H)\right\}\Bigg]_{M\to\infty} \tag{41}$$

A final remark on covariantization is that once we have done the integration over $(n,\bar{n})$, there will be terms in the action which are nonlocal in time. To go back to a Hamiltonian, one needs to remove this via the use of auxiliary fields, see [13] in this regard.

Covariantized expressions

We now show how the covariantization procedure works specifically for the mass term. As we show in the appendix, the mass term F in (15) can be written in terms of $A,\ \bar{A}$ and $\mathcal{A},\ \bar{\mathcal{A}}$ with $\bar{D}\mathcal{A} - \partial\bar{\mathcal{A}} = 0$, eqs. (A7) to (A14). Using the above equation, or eq. (A9), to express $\mathcal{A},\ \bar{\mathcal{A}}$ in terms of $\bar{A},\ A$ respectively we can write F as [2]

$$F = \frac{1}{2e^2}\int\left(A - \sum_{n=0}^{\infty}(-1)^n\frac{1}{\bar{\partial}}(\bar{A}\frac{1}{\bar{\partial}})^n\partial\bar{A}\right)^a\left(\bar{A} - \sum_{n=0}^{\infty}(-1)^n\frac{1}{\partial}(A\frac{1}{\partial})^n\bar{\partial}A\right)^a \tag{42}$$

Let us first consider the term quadratic in A's

$$F^{(2)} = \frac{1}{2e^2}\int\Big(A^a\bar{A}^a - A^a\frac{1}{\partial}\bar{\partial}A^a - \bar{A}^a\frac{1}{\bar{\partial}}\partial\bar{A}^a + \frac{1}{\bar{\partial}}(\partial\bar{A}^a)\frac{1}{\partial}(\bar{\partial}A^a)\Big) \tag{43}$$

According to the covariantization procedure outlined in section 3, we get

$$F^{(2)}_{cov} = \int d\mu\ F^{(2)}\left(A^a\to\tfrac{1}{2}n\cdot A^a,\ \bar{A}^a\to\tfrac{1}{2}\bar{n}\cdot A^a,\ \partial\to\tfrac{1}{2}n\cdot\partial,\ \bar{\partial}\to\tfrac{1}{2}\bar{n}\cdot\partial\right)$$

[2] F may also be written in terms of the magnetic field B as

$$F = \frac{1}{8e^2}\int\left(\bar{D}^{-1}\ B\right)^a\left(D^{-1}\ B\right)^a$$

$$= \frac{1}{8e^2}\int \frac{d^3k}{(2\pi)^3} A^a_\mu(-k)A^a_\nu(k)\int d\mu \left[n_\mu \bar{n}_\nu - n_\mu n_\nu \frac{\bar{n}\cdot k}{n\cdot k} - \bar{n}_\mu \bar{n}_\nu \frac{n\cdot k}{\bar{n}\cdot k} + \bar{n}_\mu n_\nu \right] \quad (44)$$

The integrals over Lorentz transformations (regularized expressions) can be evaluated as described in section 3. We have

$$\begin{aligned} \int d\mu \; n_\mu \bar{n}_\nu &= \frac{2}{3} g_{\mu\nu} \qquad\qquad \mu,\nu = 1,2,3 \\ \int d\mu \; n_\mu n_\nu \frac{\bar{n}\cdot k}{n\cdot k} &= -\frac{1}{3} g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \end{aligned} \quad (45)$$

Using (45) in (44) we get

$$F^{(2)}_{cov} = \frac{1}{4e^2}\int A^a_\mu(-k)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) A^a_\nu(k) \quad (46)$$

We now consider the term in (A10) which is cubic in A's.

$$\begin{aligned} F^{(3)} &= \frac{1}{2e^2}\int f^{abc}\Bigg(\Big[A^a \frac{1}{\partial}(A^b \frac{1}{\partial}\bar{\partial} A^c) + \bar{A}^a \frac{1}{\bar{\partial}}(\bar{A}^b \frac{1}{\bar{\partial}}\partial \bar{A}^c) \Big] \\ &\qquad - \Big[\frac{1}{\partial}\bar{\partial} A^a \frac{1}{\bar{\partial}}(\bar{A}^b \frac{1}{\bar{\partial}}\partial \bar{A}^c) + \frac{1}{\bar{\partial}}\partial \bar{A}^a \frac{1}{\partial}(A^b \frac{1}{\partial}\bar{\partial} A^c) \Big] \Bigg) \\ &\equiv F^{(3)}_{pure} + F^{(3)}_{mixed} \end{aligned} \quad (47)$$

where $F^{(3)}_{pure}$ contains only holomorphic or only antiholomorphic components of A's and $F^{(3)}_{mixed}$ contains both holomorphic and antiholomorphic components.

According to our covariantization procedure we get in momentum space,

$$\begin{aligned} F^{(3)}_{pure} &= \frac{i}{8e^2}\int f^{abc}\delta^{(3)}(p+q+k) A^a_\mu(p) A^b_\nu(q) A^c_\lambda(k) \int d\mu \left(\frac{\bar{n}\cdot k}{n\cdot p \; n\cdot k} n_\mu n_\nu n_\lambda + \text{c.c.} \right) \\ F^{(3)}_{mixed} &= \frac{i}{8e^2}\int f^{abc}\delta^{(3)}(p+q+k) A^a_\mu(p) A^b_\nu(q) A^c_\lambda(k) \int d\mu \left(\frac{\bar{n}\cdot p}{n\cdot p \; \bar{n}\cdot k} n_\mu n_\nu \bar{n}_\lambda + \text{c.c.} \right) \end{aligned} \quad (48)$$

where "c.c" denotes complex conjugation. After symmetrization over the momenta and integration over the Lorentz transformations we find

$$F^{(3)}_{pure} = -\frac{i}{8e^2}\int f^{abc}\delta^{(3)}(p+k+q) A^a_\mu(p) A^b_\nu(q) A^c_\lambda(k) \; V^{AN}_{\mu\nu\lambda}(p,q,k) \quad (49)$$

where

$$\begin{aligned} V^{AN}_{\mu\nu\lambda}(p,q,-(p+q)) &= \frac{1}{p^2q^2-(p\cdot q)^2}\Bigg[\left\{ \frac{p\cdot q}{p^2} - \frac{q\cdot(q+p)}{(p+q)^2} \right\} p_\mu p_\nu p_\lambda \\ &\quad + \frac{p\cdot(p+q)}{(p+q)^2}(q_\mu q_\nu p_\lambda + q_\lambda q_\nu p_\mu + q_\lambda q_\mu p_\nu) - (q\to p) \Bigg] \end{aligned}$$

$V^{AN}_{\mu\nu\lambda}$ is proportional to the cubic vertex appearing in the expression of the magnetic mass proposed by Alexanian and Nair in [7].

The symmetrization over momenta and Lorentz integration is a lot more involved in the case of $F^{(3)}_{mixed}$ and it was done using Mathematica. We find that

$$F^{(3)}_{mixed} = \frac{i}{24e^2}\int \delta^{(3)}(p+k+q) f^{abc} A^a_\mu(p) A^b_\nu(q) A^c_\lambda(k) \left\{ V^{AN}_{\mu\nu\lambda}(p,q,k) + L_{\mu\nu\lambda}(p,q,k) \right\} \tag{50}$$

where $L_{\mu\nu\lambda}(p,q,k)$ contains terms involving a log-dependence on the momenta.

Adding (49) and (50), we find that the total cubic order contribution of F_{cov} can be written as

$$\begin{aligned}
F^{(3)}_{cov} \quad &= -\frac{i}{12e^2}\int f^{abc}\delta^{(3)}(p+k+q) A^a_\mu(p) A^b_\nu(q) A^c_\lambda(k)\ V^{AN}_{\mu\nu\lambda}(p,q,k) \\
&- \frac{\epsilon_{\mu\nu\lambda}}{8e^2}\int \delta^{(3)}(p+k+q) f^{abc}\Bigg\{ \frac{2}{qk}\left(\frac{X}{qk - q\cdot k} + \frac{Y}{qk + q\cdot k} \right) \tilde{F}^a_\lambda(p)\tilde{F}^b_\nu(q)\tilde{F}^c_\mu(k) \\
&- \frac{2}{qk}\left(\frac{X}{(qk - q\cdot k)^2} - \frac{Y}{(qk + q\cdot k)^2} \right) \tilde{F}^a_\lambda(p)\partial_\nu\tilde{F}^b_\rho(q)\partial_\mu\tilde{F}^c_\rho(k) \\
&- \frac{4}{qk}\left(\frac{X}{(qk - q\cdot k)^2} - \frac{Y}{(qk + q\cdot k)^2} \right) \tilde{F}^a_\lambda(p)\partial_\rho\tilde{F}^b_\nu(q)\partial_\mu\tilde{F}^c_\rho(k) \\
&- \frac{4}{qk}\left(\frac{X}{(qk - q\cdot k)^3} + \frac{Y}{(qk + q\cdot k)^3} \right) \tilde{F}^a_\lambda(p)\partial_\rho\partial_\mu\tilde{F}^b_\tau(q)\partial_\nu\partial_\tau\tilde{F}^c_\rho(k) \Bigg\}
\end{aligned} \tag{51}$$

where $X = \ln\left[(qk + q\cdot k)/2qk)\right]$, $Y = \ln\left[(qk - q\cdot k)/2qk)\right]$ and $\tilde{F}^a_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}F^a_{\nu\lambda}$.

The expression (51) is true up to cubic terms in A although the log-terms were written in terms of $\tilde{F}^a_\mu$ in order to make the gauge invariance more transparent.

We see that the covariantization of F produces two series of terms: one series of terms which starts with a term quadratic in A's and higher order terms necessary for gauge invariance, and a second series of terms involving the logarithms of momenta which starts with a term cubic in A's. These two series of terms are separately gauge invariant. The non-log terms from (46) and (51) combine to give the expression for the magnetic mass term proposed in [13, 7]. Since this is essentially the covariantization of $I(H)$, we may conclude that the second series of log-terms results from the covariantization of just the WZ-term, the term cubic in $H^{-1}\partial H$, in $I(H)$. After all we have shown in the appendix, eq. (A14), that

$$F(A) = -\frac{2\pi}{e^2}\left[I(H) - \frac{i}{12\pi}\int \epsilon^{\mu\nu\alpha}\mathrm{Tr}(H^{-1}\partial_\mu H H^{-1}\partial_\nu H H^{-1}\partial_\alpha H) \right] \tag{52}$$

4. Resummation and magnetic mass

As we have stated earlier, the minimal covariantized mass term in Euclidean space agrees with what was proposed in [13]. The resummation of perturbation theory, to one-loop order with resummed propagators and vertices, was carried out in [7]. To one-loop order, Δ was obtained as $\Delta = \Delta^{(1)} \approx 1.2(e^2 c_A \mu/2\pi)$ The resulting gap equation $\Delta^{(1)} = \mu^2$ gives a value for the mass gap as $\mu \approx 1.2(e^2 c_A/2\pi)$. Considering that we are starting from a perturbative end with resummation, this value is quite close to the value $e^2 c_A/2\pi$ which we found in our Hamiltonian approach. In the light of all our discussion above, this is not so surprising because the mass term used in [7, 13] has emerged as the minimal one starting from our Hamiltonian analysis. Whether this mass term was anything special was a question raised by Jackiw and Pi in [9]. As we have seen it is a minimal, but not unique, covariant generalization of the form which emerges in the Hamiltonian analysis. In the end, the main advantage of this term might in fact be the following. Generally nonlocal vertices with covariant Green's functions can mean that there are additional propagating degrees of freedom in the theory, which may be made manifest by checking unitarity via cutting rules or by making the Lagrangian local via auxiliary fields. (The Lagrangian then has time-derivatives of the auxiliary fields which means that they are actually propagating degrees of freedom.) For the minimal term, however, the auxiliary fields have a gauged WZW action and one can argue that it has no degrees of freedom modulo the holomorphic symmetry [13]. This singles out the minimal term to some extent. Nevertheless, we are not too far from what other authors have used. Consider the nonminimal term F given in (15). Noting that the field strength $B = M^{\dagger -1}\bar{\partial} J M^\dagger = -M\partial \bar{J} M^{-1}$ and $D^{-1} = M(\partial^{-1})M^{-1}$, $\bar{D}^{-1} = M^{\dagger -1}(\bar{\partial}^{-1})M^\dagger$, we see that it is very similar to, although not exactly, $F_{\mu\nu}(\mathcal{D}^{-2})F_{\mu\nu}$, which is the form used by Jackiw and Pi in [9]. One could also go further and investigate the gap equation which results from the use of the covariantized form of F rather than F_{min}. The additional logarithmic terms in F render the calculation significantly more complicated, although there is no reason to expect the results to be dramatically different.

Now we turn to the question: how do we use this in a calculation? From a purely (2+1)-dimensional point of view, we know that there is no parameter which controls the resummed loop expansion [7, 9]. The calculation of the numerical value of the gap in this way would be difficult, at best. Our Hamiltonian approach would be better suited to such questions. One can also use the (2+1)-dimensional theory to describe magnetic screening in a quark-gluon plasma in (3+1) dimensions. Notice that one needs some perturbative gauge-invariant way of incorporating magnetic screening for the high temperature calculations with the hard thermal loop resummations used for the quark-gluon plasma. More than specific numerical values, one needs a framework for such calculations and the present work bears on this issue. (The embedding of (2+1) results in the (3+1)-dimensional theory has been discussed in [14].)

5. Discussion

A number of different concepts have been brought together in this work and it may be useful to summarize briefly what we have done. Based on our Hamiltonian analysis, one can show that there is a mass term of the form $(\partial\varphi^a \bar{\partial}\varphi^a)$ at the lowest nontrivial order in φ^a. There is no ambiguity to this order in φ^a. In generalizing from this, first of all, we need to write down an expression in terms of $H = e^{t^a \varphi^a}$ which reduces to $(\partial\varphi^a \bar{\partial}\varphi^a)$ at the lowest order. There are many such expressions. In the appendix, we outline the reasons why $(\partial\varphi^a\bar{\partial}\varphi^a)$ should be considered as the lowest order term of $\mathrm{Tr}(\partial H\bar{\partial}H^{-1})$. The argument then is to use this term, or some generalization of it, as a mass term to be used in a resummation procedure. Even at this stage, although some restrictions on the possible form of a mass term have been obtained, there are still many terms which have holomorphic invariance and agree with $\mathrm{Tr}(\partial H\bar{\partial}H^{-1})$ to the requisite order, F_{min} in Eq.(14) and F in Eq.(15) being two such expressions. F_{min} is a minimal one in the sense of not requiring additional use of Green's functions and, for this reason, leads to simpler formulae upon covariantization.

Once we have chosen a specific mass term such as F_{min}, its use in an action formalism, rather than in a Hamiltonian analysis, will require that it be covariantized to maintain Lorentz covariance order by order. We have given a method of covariantization, both in Minkowski space and in the Wick rotated Euclidean case. Finally, we have given a discussion of the results of the resummation carried out with the minimal mass term F_{min}.

APPENDIX

We have written the mass term in (12) to the second order in φ. We want to write an expression in terms of H for which this is the quadratic expansion and show that the correct expression should be $\mathrm{Tr}(\partial H\bar{\partial}H^{-1})$ and not something like $\mathrm{Tr}(\partial H\bar{\partial}H)$.

Writing the kinetic energy T as

$$T = -\frac{e^2}{2}\frac{\delta^2}{\delta A_i^a \delta A_i^a} \tag{A1}$$

we have

$$\begin{aligned}&\frac{e^2}{2}\int d\mu(H) e^{2c_A I}\Psi^*\left(-\frac{\delta^2}{\delta A_i^a \delta A_i^a}\Psi\right)\\ &= \frac{e^2}{2}\int d\mu(H)\Phi^*\left[-\frac{\delta^2\Phi}{\delta A_i^a\delta A_i^a} + 2c_A\frac{\delta I}{\delta A_i^a}\frac{\delta\Phi}{\delta A_i^a} - \left(c_A^2\frac{\delta I}{\delta A_i^a}\frac{\delta I}{\delta A_i^a} - c_A\frac{\delta^2 I}{\delta A_i^a\delta A_i^a}\right)\Phi\right]\end{aligned} \tag{A2}$$

where we have written $\Psi = e^{-c_A I}\Phi$, absorbing the crucial WZW-part of the measure into the wavefunctions. In a similar way we have

$$\begin{aligned}&\frac{e^2}{2}\int d\mu(H) e^{2c_A I}\left(-\frac{\delta^2}{\delta A_i^a \delta A_i^a}\Psi^*\right)\Psi\\ &= \frac{e^2}{2}\int d\mu(H)\left[-\frac{\delta^2\Phi^*}{\delta A_i^a\delta A_i^a} + 2c_A\frac{\delta I}{\delta A_i^a}\frac{\delta\Phi^*}{\delta A_i^a} - \left(c_A^2\frac{\delta I}{\delta A_i^a}\frac{\delta I}{\delta A_i^a} - c_A\frac{\delta^2 I}{\delta A_i^a\delta A_i^a}\right)\Phi^*\right]\Phi\end{aligned} \tag{A3}$$

We now add these two equations and do a partial integration for $\delta/\delta A_i^a$. In doing so we have to use the full measure $d\mu(H)e^{2c_A I} = [dAd\bar{A}]/(\mathrm{vol}\mathcal{G})$. This gives

$$\int d\mu(H)\frac{\delta I}{\delta A_i^a}\frac{\delta(\Phi^*\Phi)}{\delta A_i^a} \quad = \quad \int d\mu(H)e^{2c_A I}e^{-2c_A I}\frac{\delta I}{\delta A_i^a}\frac{\delta(\Phi^*\Phi)}{\delta A_i^a}$$

$$= \int d\mu(H)\left(-\frac{\delta^2 I}{\delta A_i^a \delta A_i^a} + 2c_A \frac{\delta I}{\delta A_i^a}\frac{\delta I}{\delta A_i^a}\right)\Phi^*\Phi \tag{A4}$$

Thus upon adding (A2) and (A3) and using (A4) we find

$$\langle\Psi|T|\Psi\rangle = \tfrac{1}{2}\left\langle\Phi\left|\tilde{T}+\tilde{T}^\dagger\right|\Phi\right\rangle + \frac{e^2c_A^2}{2}\left\langle\Phi\left|\frac{\delta I}{\delta A_i^a}\frac{\delta I}{\delta A_i^a}\right|\Phi\right\rangle \tag{A5}$$

where the inner product in terms of Φ's is now

$$\langle 1|2\rangle = \int d\mu(H)\Phi^*\Phi \tag{A6}$$

and $\tilde{T}\Phi = -\frac{e^2}{2}\frac{\delta^2\Phi}{\delta A_i \delta A_i}$. $\tilde{T}^\dagger$ denotes the adjoint of $\tilde{T}$ with just the Haar measure for integration as in (A6). Eq.(A5) displays the extra "mass term" as

$$\frac{e^2c_A^2}{2}\int\frac{\delta I}{\delta A_i^a}\frac{\delta I}{\delta A_i^a} = \frac{e^2c_A^2}{2}\int\frac{\delta I}{\delta A^a}\frac{\delta I}{\delta \bar{A}^a} \equiv m^2 F \tag{A7}$$

where $m = e^2c_A/2\pi$. In terms of the gauge potentials, the lowest order term of this expression, viz., the quadratic term, is the mass term (13) for A's, the higher order terms being required for reasons of gauge invariance. This term can be simplified further as follows. Regarding I as a function of A, $\bar{A}$, we can write its variation as

$$\delta I = -\frac{1}{2\pi}\int (A-\mathcal{A})^a\delta\bar{A}^a + (\bar{A}-\bar{\mathcal{A}})^a\delta A^a \tag{A8}$$

where $\mathcal{A}$, $\bar{\mathcal{A}}$ obey the equations

$$\begin{aligned}\bar{D}\mathcal{A} - \partial\bar{A} &= 0\\ D\bar{\mathcal{A}} - \bar{\partial}A &= 0\end{aligned} \tag{A9}$$

This shows that we may write

$$F = \frac{2\pi^2}{e^2}\int\frac{\delta I}{\delta A^a}\frac{\delta I}{\delta\bar{A}^a} = \frac{1}{2e^2}\int(A-\mathcal{A})^a(\bar{A}-\bar{\mathcal{A}})^a \tag{A10}$$

Taking the variation of (A10) and using (A8) we find

$$-\frac{e^2}{\pi}\delta F = \delta I(A,\bar{A}) - \frac{1}{2\pi}\int(\mathcal{A}-A)^a\delta\bar{\mathcal{A}}^a + (\bar{\mathcal{A}}-\bar{A})^a\delta\mathcal{A}^a \tag{A11}$$

Notice that the second term is just like the variation of I as in (A8), except for the exchange $\mathcal{A}\to A$, $\bar{\mathcal{A}}\to\bar{A}$. In terms of the parametrization (1) of A, $\bar{A}$, we can solve (A9) to get

$$\begin{aligned}\bar{\mathcal{A}} &= -\bar{\partial}MM^{-1} = M\bar{\partial}M^{-1}\\ \mathcal{A} &= M^{\dagger-1}\partial M^\dagger = -\partial M^{\dagger-1}M^\dagger\end{aligned} \tag{A12}$$

The exchange $\mathcal{A} \to A$, $\bar{\mathcal{A}} \to \bar{A}$ thus corresponds to $M \to M^{\dagger -1}$ or $H = M^\dagger M \to H^{-1} = M^{-1} M^{\dagger -1}$. Equation (A11) can thus be written as

$$-\frac{e^2}{\pi}\delta F = \delta I(H) + \delta I(H^{-1}) \tag{A13}$$

This implies

$$F = -\frac{\pi}{e^2}\left[I(H) + I(H^{-1})\right] = -\frac{1}{e^2}\mathrm{Tr}(\bar{\partial} H \partial H^{-1}) \tag{A14}$$

This brings us to the point of identifying the mass term which satisfies the requirements 1 and 2 listed in section 2, but not yet the requirement 3. The expression for F as it is written in (A14) is not holomorphically invariant. This is because, eventhough $I(H)$ is invariant, $I(H^{-1})$ is not. (Notice that the inversion of D, $\bar{D}$ to obtain $\mathcal{A}$, $\bar{\mathcal{A}}$, or equivalently the solution (A12), requires fixing a "holomorphic frame". This is why the form of F in (A14) is not holomorphically invariant.) A holomorphically invariant completion of F is straightforward. Notice that F in (A14) is proportional to the kinetic term of $I(H)$. Since $I(H)$ is invariant under $H \to V H \bar{V}$, we see that a minimal completion of F we can use is

$$F \to F_{min} = -\frac{2\pi}{e^2} I(H) \tag{A15}$$

The minimal mass term is then the WZW action.

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